Model kinetic equation for low-density granular flow

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(Received 6 October 1995; revised manuscript received 26 February 1996)

A model kinetic equation is proposed to describe the time evolution of a gas composed of particles which collide inelastically. Dissipation in collisions is described by means of a parameter which is related to the coefficient of restitution, ϵ. The kinetic equation can be solved exactly for the homogeneous cooling state, providing explicit expressions for both the time-dependent ''temperature'' and the velocity distribution function. In contrast to the Maxwellian for fluids with energy conservation, this distribution exhibits algebraic decay for large velocities. Hydrodynamic equations are derived by expanding in the gradients of the hydrodynamic fields around the homogeneous cooling state, without the limitation to ϵ asymptotically close to unity. The equation for the energy density contains, in addition to a source term describing the energy lost in collisions, a contribution to the heat flux which is proportional to the gradient of the density. The linear stability of the homogeneous cooling state is investigated by analyzing the hydrodynamic modes of the system. The shear modes are found to decay slowly at long wavelengths, in the sense that spatial perturbations of the macroscopic flow field decay slower than the cooling rate for the thermal velocity of the reference homogeneous state. On the other hand, the heat mode is always stable. [S1063-651X(96)02107-1]

PACS number(s): 05.20.Dd, 05.60.+w, 47.20.−k, 47.50.+d

I. INTRODUCTION

The study of fluids composed of inelastically colliding particles has attracted a lot of attention in the last few years. This interest has been prompted and stimulated by the attempts to understand the motion of granular media in the so-called rapid flow regime, in which the particles move freely and independently except when they collide with each other. In this regime, one can think of the grains of the granular material as similar to the molecules of a fluid and try to extend the methods of kinetic theory in order to describe the behavior of the medium. This analogy has been used by several authors to derive continuum hydrodynamic-like equations [1] . In some cases the starting level of description is purely microscopic, while in others physical arguments are introduced to justify the form of the fluid equations. Nevertheless, all theories arrive at roughly the same type of equations. They are similar to the conventional Navier-Stokes equations, the main difference being the presence of a term describing dissipation of macroscopic kinetic energy into thermal energy in the evolution equation for the temperature. The macroscopic description of the flow is made just as for ordinary fluids in terms of the density, the macroscopic flow velocity, and the granular temperature, although the nature of the latter is quite different from the thermodynamic temperature of a fluid [1]. It is worth mentioning that recent work on one-dimensional granular media has incorporated, as an additional hydrodynamic field, the third moment of the fluctuating velocity [2].

The above equations of motion have been used to analyze the linear stability of granular flows, focusing the attention on the possibility of explaining the formation of the ''inelastic structures'' or clusters which have been observed in computer simulations [3–5]. The stability of two particular states corresponding to different physical situations has been investigated: the unbounded uniform shear flow [6,7] and the homogeneous ''basic'' state [5,7]. The latter is characterized by a vanishing flow field and uniform density and temperature. In contrast to normal fluids, the temperature decreases monotonically and consequently this state is referred to also as the ''cooling'' granular state. In the two works we are aware of which deal with the stability of cooling granular media [5,7] both use the same constitutive relations (i.e., expressions for the heat and momentum fluxes in terms of hydrodynamic gradients) derived by Jenkins and Richman [8].

Although a hydrodynamic description for systems with inelastic collisions is suggested by analogy with normal fluids, its form and justification requires a detailed derivation of the equations from a more fundamental basis. The kinetic model proposed here provides a simple but more fundamental level of description from which the context of a hydrodynamic description can be addressed exactly. For normal fluids, hydrodynamics represents the dominant phenomena at long wavelengths and long times. This dominance is assured by the fact that the hydrodynamic fields are associated with the five globally conserved quantities (particle number, total energy, and total momentum) and the fact that the asymptotic state is both uniform and stationary. In the present case energy is not conserved and consequently the asymptotic state is not stationary. It is shown that a universal solution is approached for a wide class of spatially homogeneous initial conditions, whose time dependence occurs entirely through the temperature field. The latter can be calculated directly from the kinetic equation and sets the time scale for approach to this universal homogeneous cooling state. This asymptotic state therefore plays the same role for fluids with

inelastic collisions as the Maxwellian does for normal fluids. In the latter case, a local equilibrium generalization of the Maxwellian serves as the reference state for study of inhomogeneous states. A similar *local* cooling state is established here as well. Hydrodynamics follows from the kinetic equation if there exist ''normal'' solutions whose space and time dependence occurs only through the hydrodynamic fields. Such a solution is constructed perturbatively, by expanding about the local cooling state in powers of the gradients of the hydrodynamic fields. This is a generalization of the Chapman-Enskog method for deriving hydrodynamics from the Boltzmann equation, with the important difference that the reference state can be very far from the Maxwellian. Therefore, in contrast to previous studies along these lines, the assumption of small spatial gradients does not imply weak dissipation and the hydrodynamics represents a description of spatial excitations relative to this dynamic state. For strong dissipation the reference state can change on time scales comparable to the relaxation times for small spatial perturbations. Since the asymptotic reference state is spatially uniform, the hydrodynamic description dominates over all other excitations at long times since their relaxation times increase with the wavelength of the disturbance.

Because the reference state is not Maxwellian, there is no reason to expect that the constitutive relations between the average heat and momentum fluxes are given simply by Newton's and Fourier's laws, respectively. In fact, we show that the model kinetic equation proposed here leads to an additional term in the heat flux proportional to the density gradient. Such a term has been obtained by Lun *et al.* [9], but they restrict their results to asymptotically weak dissipation. This precludes study of the qualitative effects of this term described in Sec. II under conditions of strong dissipation. The Chapman-Enskog expansion in [9] uses a Maxwellian as the reference state and therefore does not predict a limit on the dissipation due to a slow decay of the homogeneous cooling state distribution. In particular, it is shown that the contribution to the heat flux from a density gradient stabilizes the longitudinal shear mode at large dissipation. Since stability plays a central role in the analysis of possible states for granular flow, and since experiments and also computer simulations have shown that highly nonlinear effects dominate the behavior of granular media, it is essential to have a secure basis for the hydrodynamic equations used in their analysis. For all these reasons, it is instructive to study simple model kinetic equations of inelastic gases which allow controlled and detailed analysis. Although these models do not provide a quantitative description for real granular media, they can isolate and clarify some important qualitative features of the physics peculiar to granular flows.

The model presented here is an extension to dissipative gases of the well-known Bhatnagar-Gross-Krook (BGK) model equation [10], which has proved to be very useful for the study of both stationary and nonstationary states of normal fluids far from equilibrium [11]. There is not a unique modification of this equation to take account of dissipation in collisions, but an argument is presented in Appendix A to support a particularly simple possibility. The resulting equation retains all of the qualitative practical advantages of the BGK equation for normal fluids. Attention is limited here to the homogeneous cooling state and to those with small spatial deviations from this state. However, a significant advantage of the BGK model is that more complex conditions (e.g., boundary driven) can be studied quantitatively. The primary results obtained here are (1) the velocity distribution for the homogeneous cooling state is broad, with divergent moments of degree $k \geq 2/(1-\epsilon^2)$; (2) a hydrodynamic description exists at Navier-Stokes order for $\epsilon^2 > 1/2$; (3) the momentum flux is given by Newton's viscosity law, while the heat flux is given by Fourier's law plus an additional contribution proportional to the density gradient; (4) the linearized hydrodynamic equations yield shear modes that decay slower than the cooling rate of the reference homogeneous state.

The plan of the paper is as follows. The model is formulated in Sec. II. The balance equations of average density, momentum, and energy are obtained from moments of the kinetic equation, and the exact solutions of the balance equations and the kinetic equation for the homogeneous cooling state are obtained. In Sec. III the generalized Chapman-Enskog expansion for inhomogeneous normal solutions is carried out to first order in the gradients (Navier-Stokes order). The result is used to calculate the heat and momentum fluxes, and associated transport coefficients as explicit functions of ϵ. In Sec. IV the stability of the time-dependent homogeneous cooling state is considered by linearizing the hydrodynamic equations around it. Some comments on the comparison of our results with previous studies are given there as well. Finally, Sec. V provides a summary and conclusions.

II. THE KINETIC MODEL EQUATION

In a low density gas mass, momentum, and energy conserving collisions tend to produce a local Maxwellian distribution of velocities. A model kinetic equation with these properties is given by one whose change due to collisions is simply proportional to the deviation of the distribution function from this local distribution. The parameters of the local distribution (local density, temperature, and flow velocity) are determined such that the conservation laws hold. Here we want to extend this model to admit loss of energy during the collision process. The simplest way is to scale the temperature in the local distribution by a parameter measuring the energy loss. Thus we propose the following equation for the one particle distribution function $f(\mathbf{r},\mathbf{v},t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f = -\zeta(f - f_0), \tag{1}$$

where $\zeta(\mathbf{r},t)$ is an average collision frequency (specified in more detail below), and

$$f_0(\mathbf{r},\mathbf{v},t) = n(\mathbf{r},t)\,\phi[V/\epsilon v_0(\mathbf{r},t)]. \tag{2}$$

Here $\mathbf{V}(\mathbf{r},t) = \mathbf{v} - \mathbf{u}(\mathbf{r},t)$ is the ''peculiar'' velocity, $v_0(\mathbf{r},t) = [2k_B T(\mathbf{r},t)/m]^{1/2}$ is the average thermal velocity (k_B is Boltzmann's constant, m is the particle mass), and $\phi(v/\sigma)$ is the normalized Maxwellian for d dimensions,

$$\phi(v/\sigma)=(\pi\sigma^2)^{-d/2}\exp\left[-\left(\frac{v}{\sigma}\right)^2\right]. \tag{3}$$

The functions n, $\mathbf{u}$, and T are identified as the local particle number density, flow velocity, and temperature, respectively, through the definitions

$$n(\mathbf{r},t)=\int d\mathbf{v}\, f(\mathbf{r},\mathbf{v},t), \tag{4a}$$

$$n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)=\int d\mathbf{v}\,\mathbf{v} f(\mathbf{r},\mathbf{v},t), \tag{4b}$$

$$\frac{d}{2}n(\mathbf{r},t)k_BT(\mathbf{r},t)=\int d\mathbf{v}\frac{1}{2}mV^2 f(\mathbf{r},\mathbf{v},t). \tag{4c}$$

It is customary in the literature of granular media to define the ''granular'' temperature by Eq. (4c) without the Boltzmann constant. Since our model is formulated in the general context of kinetic theory of dissipative gases we prefer to keep the standard definition in kinetic theory. Finally, ϵ is a position- and time-independent constant with $0<\epsilon\leqslant 1$. As shown below, it represents the coefficient of restitution for inelastic collisions. For $\epsilon=1$, Eq. (1) reduces to the BGK model kinetic equation [10] for normal gases.

The context of this choice for the BGK kinetic model is discussed in Appendix A. Some comments on its relevance here are appropriate. For elastic collisions, the right side of Eq. (1) describes a detail balance condition whereby the collisions drive the system towards a local equilibrium distribution. In the presence of external forces or boundary conditions, there is a competition between this collisional detail balance and the external constraints, leading to possible stationary distributions quite different from local equilibrium. In the case of inelastic collisions detailed balance is violated even in the absence of external constraints, and collisions do not stabilize the local equilibrium distribution. This effect is represented by the dependence of f_0 on ϵ in Eq. (1). It does not mean, however, that inelastic collisions stabilize f_0 instead. The fact that f_0 cannot be a solution follows from the violation of energy conservation and a consequent time dependence of f_0 through its dependence on temperature. This is apparent in the detailed derivation of the homogeneous cooling state, f_h, given below.

Multiplication of Eq. (1) by 1, $\mathbf{v}$, and v^2 and integration over the velocity leads to the following evolution equations for the hydrodynamic fields:

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot(n\mathbf{u})=0, \tag{5}$$

$$mn\frac{\partial \mathbf{u}}{\partial t}+mn\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=-\nabla\cdot\mathbf{P}, \tag{6}$$

$$\frac{d}{2}nk_B\frac{\partial T}{\partial t}+\frac{d}{2}nk_B\mathbf{u}\cdot\boldsymbol{\nabla}T$$

$$=-(\boldsymbol{\nabla}\mathbf{u}):\mathbf{P}-\boldsymbol{\nabla}\cdot\mathbf{q}-\frac{d}{2}\zeta(1-\epsilon^2)nk_BT, \tag{7}$$

where

$$\mathbf{P}(\mathbf{r},t)=\int d\mathbf{v}\, m\mathbf{V}\mathbf{V}f(\mathbf{r},\mathbf{v},t) \tag{8}$$

is the pressure tensor, and

$$\mathbf{q}(\mathbf{r},t)=\int d\mathbf{v}\frac{1}{2}m\mathbf{V}^2\mathbf{V}f(\mathbf{r},\mathbf{v},t) \tag{9}$$

is the heat flux. The above equations are similar to the conservation laws for a dilute gas with elastic collisions except for the presence of the additional energy dissipation term proportional to $(1-\epsilon^2)$ in the temperature equation. From the structure of this term it is seen that $1-\epsilon^2$ is the average of the relative loss of kinetic energy per particle in each collision. Therefore ϵ can be interpreted as the coefficient of restitution of collisions [1]. This is confirmed by comparing the above equation with the exact balance equation for the energy derived from the Boltzmann equation for a granular gas.

The form of ζ can be made more explicit by identifying it as the average collision frequency for the Boltzmann gas. Only the case of hard sphere interactions is considered, for which ζ has the form

$$\zeta(\mathbf{r},t)=Cn(\mathbf{r},t)T(\mathbf{r},t)^{1/2}, \tag{10}$$

where C is a constant depending on the dimension of the system.

Consider now the basic homogeneous solution of Eqs. (5)–(7) defined by the conditions

$$\mathbf{u}_h=0,\quad \boldsymbol{\nabla}T_h=\boldsymbol{\nabla}n_h=0,\quad \boldsymbol{\nabla}\cdot\mathbf{q}_h=0, \tag{11}$$

which imply

$$\frac{\partial n_h}{\partial t}=0,\quad \boldsymbol{\nabla}\cdot\mathbf{P}_h=0, \tag{12}$$

and

$$\frac{\partial T_h}{\partial t}=-\zeta_h(1-\epsilon^2)T_h. \tag{13}$$

Due to the dissipation in collisions this homogeneous state is not time independent, but its temperature monotonically decreases, an effect that is usually referred to as ''cooling.'' Inserting Eq. (10) into Eq. (13) leads to a closed differential equation that can be integrated. The solution is

$$T_h(t)=\frac{T_h(0)}{\left(1+\frac{t}{t_0}\right)^2}, \tag{14}$$

where $t_0^{-1}=(1-\epsilon^2)\zeta_h(0)/2$, $\zeta_h(0)=\zeta(n_h,T_h(0))$, and $T_h(0)$ is the initial temperature. This is a well-known result in the granular flow literature and has been derived by different methods [4,12], and molecular dynamics simulations have shown that it is obeyed at short times for values of ϵ close to one (quasielastic limit).

The velocity distribution function for this homogeneous state follows from Eq. (1),

$$\frac{\partial f_h(\mathbf{v},t)}{\partial t} = -\zeta_h(t)[f_h(\mathbf{v},t) - f_{0,h}(\mathbf{v},t)]. \tag{15}$$

The distribution $f_{0,h}$ is obtained from Eq. (2) by using the hydrodynamic fields Eq. (11). The solution to this equation is obtained in Appendix B. For a general homogeneous initial distribution specified at $t=0$ the solution approaches at long times a universal distribution whose time dependence occurs entirely through $T(t)$. The universal distribution is given by

$$f_h(\mathbf{v},t) = \int_1^\infty dx\, P(x) n_h \phi(v/v_0 \epsilon x), \tag{16}$$

$$P(x) = p x^{-(1+p)}, \quad p = p(\epsilon) = \frac{2}{1-\epsilon^2}, \tag{17}$$

where $v_0 = v_0(t)$ is the thermal velocity [see following Eq. (2)] at the temperature $T_h(t)$. The distribution function f_h plays the same role for granular fluids as does the Maxwellian for normal fluids. However, there are qualitative differences in the velocity dependence since f_h is a superposition of Maxwellians including those with arbitrarily large half-widths. The distribution determining these half-widths, $P(x)$, decreases slowly for large x so there are significant contributions from large velocities. For fixed, finite ϵ, the asymptotic behavior of the distribution function can be obtained directly from Eqs. (16) and (17) with the result

$$f_h(\mathbf{v},t) \to \frac{n_h}{2} p \Gamma\left(\frac{p+d}{2}\right) (\epsilon^2 v_0^2 \pi)^{-d/2} \left(\frac{\epsilon v_0}{v}\right)^{p+d}. \tag{18}$$

As a consequence of this algebraic decay, velocity moments of degree $\geqslant p(\epsilon)$ are divergent. This result holds independent of the dimension. For $\epsilon \sim 1$ the distribution $P(x)$ is sharply peaked about $x=1$ and the velocity distribution f_h approaches the Maxwellian for normal fluids.

III. HYDRODYNAMIC EQUATIONS

The balance equations, Eqs. (5)–(7), are not a closed set of equations for the hydrodynamic fields until ''constitutive equations'' are obtained for the pressure tensor and heat flux, expressing them also in terms of the fields. In this section we derive approximate constitutive equations which are valid to first order in small gradients of the fields. These results, together with the balance equations, are analogous to the Navier-Stokes hydrodynamics for normal fluids. We use an extension of the Chapman-Enskog method [13] which accounts for the peculiarities associated with nonconservation of the energy and a resulting reference state that differs from the local equilibrium state for normal fluids. The distribution function for the hydrodynamic state is normal, i.e., it depends on space and time only through the hydrodynamic fields. Consequently, for weakly inhomogeneous states it can be expanded in powers of the gradients of these fields (formally measured by a uniformity parameter ϑ associated with each gradient operator),

$$f(\mathbf{r},\mathbf{v},t) = f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) + \vartheta f^{(1)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) + \cdots, \tag{19}$$

where $\{\chi_i(\mathbf{r},t)\}$ denotes the hydrodynamic fields, $n(\mathbf{r},t)$, $T(\mathbf{r},t)$, and $\mathbf{u}(\mathbf{r},t)$. Substitution of this expansion into Eqs. (8) and (9) gives the pressure tensor and the heat flux as an expansion in powers of the gradients. Since the coefficients are averages over normal distributions, these fluxes are expressed entirely in terms of the hydrodynamic fields and their use in the balance equations, Eqs. (5)–(7), gives a closed set of hydrodynamic equations.

Terms proportional to ϑ measure deviations from $f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\})$ at each instant in time. Thus while the reference state may evolve rapidly at strong dissipation through its dependence on $\{\chi_i(\mathbf{r},t)\}$, these deviations ''track'' this background evolution as well through their dependence on these same fields. Similarly, the parameters of the hydrodynamic equations (e.g., transport coefficients) will have a dependence on the time varying fields $\{\chi_i(\mathbf{r},t)\}$. This is the case as well for the usual Chapman-Enskog expansion about the local equilibrium state for normal fluids. The difference here is a new time scale for these fields set by the rate of dissipation. Since it is independent of the wavelength, the relaxation of the hydrodynamic modes at long wavelength can be slow relative to the reference state dynamics at strong dissipation. However, this interesting effect does not invalidate the modified Chapman-Enskog method, which requires only that small relative spatial perturbations remain small. The latter is the requirement for dynamical stability of the reference state discussed in the next section.

To determine the coefficients in the gradient expansion for the fluxes, Eq. (19) is substituted into the model kinetic equation. In addition to terms proportional to gradients of the hydrodynamic fields the time derivative of these fields occurs as well. These time derivatives can be expressed formally in terms of the gradients using the hydrodynamic equations. This implies a corresponding expansion for the time derivative in powers of the uniformity parameter

$$\frac{\partial}{\partial t} = \frac{\partial^{(0)}}{\partial t} + \vartheta \frac{\partial^{(1)}}{\partial t} + \cdots. \tag{20}$$

The expansions (19) and (20) allow a self-consistent determination of both the distribution function and the fluxes at each order in the uniformity parameter. A primary difference from the Chapman-Enskog expansion for normal fluids is that the contribution from $\partial^{(0)}/\partial t$ is nonzero due to the source term in the energy equation.

The details of this generalized Chapman-Enskog expansion for granular fluids are carried out in Appendix C. The distribution function to zeroth order in the gradients is found to be

$$f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) = \int_1^\infty dx\, P(x) n(\mathbf{r},t) \phi(V/v_0 \epsilon x), \tag{21}$$

where $v_0 = v_0(\mathbf{r},t)$ is now the thermal velocity determined from the nonequilibrium temperature $T(\mathbf{r},t)$. This is a superposition of *local* Maxwellians, and represents the extension

of the globally homogeneous distribution, Eq. (16), to its local form. The contribution from first order in the gradients is

$$f^{(1)}(\mathbf{v};|\{\chi_i(\mathbf{r},t)\})=\mathbf{A}_n(\mathbf{V},\{\chi_j(\mathbf{r},t)\})\cdot\nabla n(\mathbf{r},t)+\mathbf{A}_{\mathbf{u}}(\mathbf{V},\{\chi_j(\mathbf{r},t)\}):\nabla\mathbf{u}(\mathbf{r},t)+\mathbf{A}_T(\mathbf{V},\{\chi_j(\mathbf{r},t)\})\cdot\nabla T(\mathbf{r},t). \tag{22}$$

First order differential equations determining the functions $\mathbf{A}_n,\mathbf{A}_{\mathbf{u}}$, and $\mathbf{A}_T$ can be found in Appendix C where it is verified also that the usual solution for normal fluids is regained for $\epsilon=1$. However, the term proportional to $\nabla\mathbf{n}(\mathbf{r},t)$ is present only for $\epsilon\neq 1$ and represents a qualitative difference between granular and normal fluids. Further analysis of the consequences of this term is given in the next section.

The pressure tensor and heat flux can be calculated directly to first order in the gradients using Eqs. (21) and (22) in Eqs. (8) and (9), with the results

$$\mathbf{P}(\mathbf{r},t)\to nk_BT\mathbf{I}-\eta\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{+}-\frac{2}{d}\mathbf{I}\nabla\cdot\mathbf{u}\right), \tag{23}$$

$$\mathbf{q}(\mathbf{r},t)\to-\kappa\nabla T-\mu\nabla n. \tag{24}$$

The equation for the pressure tensor has the same form as that for a normal low density gas in the Navier-Stokes approximation (i.e., Newton's viscosity law). The first term on the right side of the equation for the heat flux corresponds to Fourier's law in normal hydrodynamics, while the second term reflects a coupling between density gradient and heat flux due to the inelastic collisions and has no analog in normal fluids. The shear viscosity coefficient η is

$$\eta=\frac{2nk_BT}{(1+\epsilon^2)\zeta}. \tag{25}$$

The heat conductivity κ and the new transport coefficient μ are

$$\kappa=\frac{(d+2)(2\epsilon^4-2\epsilon^2+1)nk_B^2T}{2(2\epsilon^2-1)^2m\zeta}, \tag{26}$$

$$\mu=\frac{(d+2)\epsilon^2(1-\epsilon^2)(k_BT)^2}{(3\epsilon^2-1)(2\epsilon^2-1)^2m\zeta}. \tag{27}$$

In the limit $\epsilon\to 1$ the coefficient μ vanishes while η and κ reduce to the shear viscosity and heat conductivity obtained from the BGK model kinetic equation [14].

Finally, combining these fluxes with the balance equations gives the Navier-Stokes order hydrodynamic equations:

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{u})=0, \tag{28a}$$

$$mn\frac{\partial\mathbf{u}}{\partial t}+mn\mathbf{u}\cdot\nabla\mathbf{u}=\nabla\cdot\left[\eta\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{+}-\frac{2}{d}\mathbf{I}\nabla\cdot\mathbf{u}\right)\right]-\nabla(nk_BT), \tag{28b}$$

$$\frac{d}{2}nk_B\frac{\partial T}{\partial t}+\frac{d}{2}nk_B\mathbf{u}\cdot\nabla T=-\frac{d}{2}\zeta n(1-\epsilon^2)k_BT+(\nabla\mathbf{u}):\left[\eta\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{+}-\frac{2}{d}\mathbf{I}\nabla\cdot\mathbf{u}\right)\right]+\nabla\cdot(\kappa\nabla T)+\nabla\cdot(\mu\nabla n)-nk_BT\nabla\cdot\mathbf{u}. \tag{28c}$$

The detailed derivation in Appendix C shows that these results hold only for $\epsilon^2>1/2$; otherwise the contributions of first order in the gradients to the heat flux diverge. It is possible that when higher orders in the gradients are considered the divergence moves to larger values of ϵ, indicating that the Chapman-Enskog expansion is only asymptotic for any value of ϵ^2 greater than 1/2. In either case, convergent or asymptotic, this establishes the maximum value of the dissipation in collisions which is compatible with Navier-Stokes order hydrodynamics according to our kinetic model. It is tempting to speculate that this result might be associated with the phenomenon of inelastic collapse which has been observed in molecular dynamics simulation of a two-dimensional gas of inelastic disks [15]. A similar effect had been previously documented in one dimension [4,16]. The limitation found here, $\epsilon\simeq 0.7$, is close to the value for which the collapse is observed in Ref. [15]. Nevertheless, we do not have any solid physical reason to relate both results.

IV. STABILITY OF THE COOLING HOMOGENEOUS SOLUTION

According to the analysis of Sec. II the Navier-Stokes equations, Eqs. (28), admit the basic homogeneous solution defined by Eqs. (11)–(13). Here we use the Navier-Stokes order equations to study the linear stability of the homogeneous state. Small perturbations of the hydrodynamic fields about this state are defined by

$$\delta n(\mathbf{r},t)=n(\mathbf{r},t)-n_h,\quad \delta\mathbf{u}(\mathbf{r},t),$$
$$\delta T(\mathbf{r},t)=T(\mathbf{r},t)-T_h(t). \tag{29}$$

Substitution of Eqs. (29) into Eqs. (28) and linearization in the perturbations gives

$$\frac{\partial}{\partial t}\delta n+n_h\nabla\cdot\delta\mathbf{u}=0, \tag{30}$$

$$mn_h\frac{\partial}{\partial t}\delta\mathbf{u}+n_hk_B\nabla\delta T+T_hk_B\nabla\delta n-\eta_h\left(\nabla^2\delta\mathbf{u}+\frac{d-2}{d}\nabla(\nabla\cdot\delta\mathbf{u})\right)=0, \tag{31}$$

$$\frac{d}{2}n_hk_B\frac{\partial}{\partial t}\delta T+\frac{3d}{4}(1-\epsilon^2)n_hk_B\zeta_h\delta T+\frac{d}{2}(1-\epsilon^2)k_BT_h\zeta_h\delta n+n_hk_BT_h\nabla\cdot\delta\mathbf{u}-\kappa_h\nabla^2\delta T-\mu_h\nabla^2\delta n=0. \tag{32}$$

Here n_h is chosen to be the initial average number density

$$n_h=\frac{1}{V}\int d\mathbf{r}\, n(\mathbf{r},0) \tag{33}$$

and $T_h(t)$ is the solution of Eq. (13) with the initial condition

$$T_h(0)=\frac{1}{N}\int d\mathbf{r}\, n(\mathbf{r},0)T(\mathbf{r},0). \tag{34}$$

In the above expressions N and V are the number of particles and the volume of the system, respectively.

These linear equations have time-dependent coefficients. However, with a suitable change of variables they can be transformed to linear equations with constant coefficients. First, dimensionless time and lengths are introduced by

$$\tau=\int_0^t dt'\,\zeta_h(t'), \quad \mathbf{l}=\frac{\zeta_h(t)}{v_0(t)}\mathbf{r}. \tag{35}$$

The physical meaning of τ is the cumulative number of collisions suffered by a particle in the interval $(0,t)$. The space scaling is time independent for the case of hard spheres we are considering. Finally, dimensionless hydrodynamic field perturbations are defined by

$$\theta(\mathbf{l},\tau)=\frac{\delta T(\mathbf{r},t)}{T_h(t)}, \quad \mathbf{w}(\mathbf{l},\tau)=\frac{\delta\mathbf{u}(\mathbf{r},t)}{v_0(t)},$$

$$\rho(\mathbf{l},\tau)=\frac{\delta n(\mathbf{r},t)}{n_h}. \tag{36}$$

In terms of these new variables the linearized hydrodynamic equations have constant coefficients and read

$$\frac{\partial\rho}{\partial\tau}+\boldsymbol{\nabla}_l\cdot\mathbf{w}=0, \tag{37a}$$

$$\frac{\partial\mathbf{w}}{\partial\tau}-\frac{1}{2}(1-\epsilon^2)\mathbf{w}+\frac{1}{2}\boldsymbol{\nabla}_l(\theta+\rho)$$

$$-\frac{1}{\epsilon^2+1}\left(\nabla_l^2\mathbf{w}+\frac{d-2}{d}\boldsymbol{\nabla}_l(\boldsymbol{\nabla}_l\cdot\mathbf{w})\right)=0, \tag{37b}$$

$$d\frac{\partial\theta}{\partial\tau}+d(1-\epsilon^2)\left(\frac{1}{2}\theta+\rho\right)+2\boldsymbol{\nabla}_l\cdot\mathbf{w}$$

$$-\nabla_l^2(\kappa^*\theta+\mu^*\rho)=0, \tag{37c}$$

where we have introduced

$$\kappa^*(\epsilon)=2\zeta_h(t)\kappa(\epsilon)/[n_h k_B v_0^2(t)],$$

$$\mu^*(\epsilon)=2\zeta_h(t)\mu(\epsilon)/[k_B T_h(t) v_0^2(t)]. \tag{38}$$

The dependence of κ and μ on ϵ is given in Eqs. (26) and (27).

To identify the linear modes and dispersion relations of Eqs. (37), we look for solutions of the form

$$\rho=\hat{\rho}(s,\mathbf{k})e^{s\tau+i\mathbf{k}\cdot\mathbf{l}}, \tag{39}$$

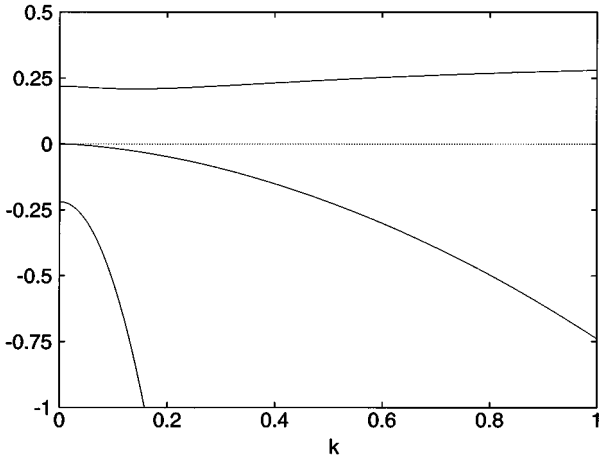

FIG. 1. Hydrodynamic modes given by the dispersion relation in Eq. (42), with $\epsilon=0.75$. The three modes are real (nonpropagating), and the longitudinal shear mode is unstable for all values of the wave-number k.

and similar expressions for $\mathbf{w}$ and θ. Since τ is a monotonically increasing function of t, the condition for stability is $\mathrm{Re}s<0$. The equations which govern the components of the velocity perpendicular to the wave-number vector $\mathbf{k}$ decouple from the rest of the equations and represent $d-1$ degenerate shear modes, with the dispersion relation

$$s_{\mathrm{trans}}=\frac{1}{2}(1-\epsilon^2)-\frac{k^2}{\epsilon^2+1}. \tag{40}$$

There shear modes become positive for $k\leqslant k^*_{\mathrm{shear}}$, where

$$k^*_{\mathrm{trans}}=\left(\frac{1-\epsilon^4}{2}\right)^{1/2}. \tag{41}$$

However, this does not mean that velocity perturbations $\delta\mathbf{u}$ grow in time since they have been scaled to the thermal velocity of the homogeneous state in the above analysis. Instead it shows that $\delta\mathbf{u}$ decays more slowly than this thermal velocity, i.e., the cooling rate is greater than the relaxation of the shear modes. After an initial transient period the linear analysis is no longer valid because terms nonlinear in the macroscopic velocity become important. This effect has been analyzed by Goldhirsch and Zanetti [5] and used as the starting point to propose a physical mechanism for the formation of high density clusters.

The other three modes are given by the solutions of the dispersion relation

$$s^3+\frac{\kappa^*(1+\epsilon^2)+2(d-1)}{d(1+\epsilon^2)}k^2s^2-\left(\frac{1}{4}(\epsilon^2-1)^2\right.$$

$$\left.-\frac{3d-\kappa^*+4\epsilon^2-d\epsilon^2+\kappa^*\epsilon^4}{2d(1+\epsilon^2)}k^2-\frac{2\kappa^*(d-1)}{d^2(1+\epsilon^2)}k^4\right)s$$

$$-\frac{1-\epsilon^2}{4}k^2+\frac{\kappa^*-\mu^*}{2d}k^4=0. \tag{42}$$

Figures 1–4 show this dispersion relation for a two-dimensional system and four values of ϵ. Only the real parts

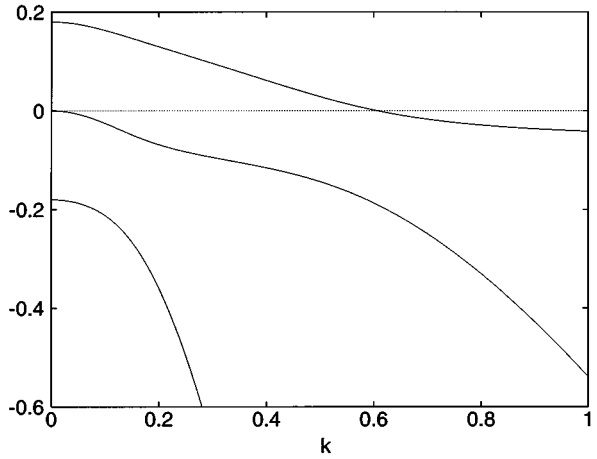

FIG. 2. The same as in Fig. 1, but for $\epsilon=0.80$. The three modes are still real in all the plotted range of k, but now the longitudinal shear mode becomes stable for $k>k^*_{\rm long}\simeq 0.6$.

of the propagating modes have been plotted. Let us focus on the leading eigenvalue, which is always real and positive for small enough values of k . It is easily identified as the extension to granular fluids of the longitudinal shear of normal fluids, and we represent it by $s_{\rm long}$. Comparison of the several curves indicates that there is an important qualitative change in the behavior of this mode at a given value ϵ^*. For $\epsilon<\epsilon^*$ it remains positive for all values of the wave number k, while for $\epsilon>\epsilon^*$ it becomes negative for k larger than a critical value $k^*(\epsilon)$. The value of ϵ^* can be obtained from the $k\to\infty$ behavior of Eq. (42). In this limit there is a real solution given by

$$s_{\rm long}(k\to\infty)=-\frac{d(1+\epsilon^2)(\kappa^*-\mu^*)}{4\kappa^*(d-1)}. \tag{43}$$

The sign of the right-hand side of this expression is determined by the factor $\kappa^*-\mu^*$. Using Eqs. (26) and (27), one gets that $\kappa^*-\mu^*>0$ for $0.626\lesssim\epsilon^2\leqslant 1$, and $\kappa^*-\mu^*<0$ for $0.5<\epsilon^2\lesssim 0.626$. Therefore $\epsilon^*\simeq 0.79$. We note that this value

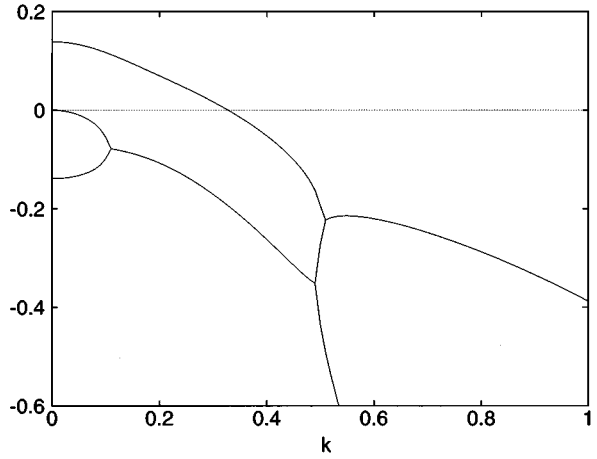

FIG. 3. The same as in Fig. 1, but for $\epsilon=0.85$. Now there are intervals of values of k in which two of the modes are complex, i.e., propagating. They correspond to the standard sound waves. Only their real part is plotted.

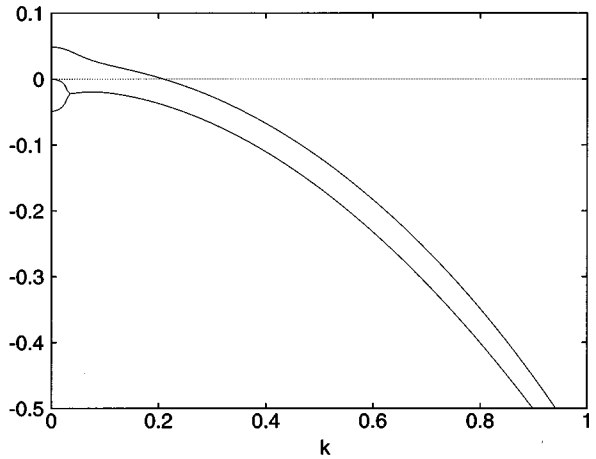

FIG. 4. The same as in Fig. 1, but now for $\epsilon=0.95$. In this case $k^*_{\rm long}\simeq 0.21$, and the sound propagating modes appear for $k\simeq 0.03$. Notice that the short wave-number part of the spectrum simplifies as the value of ϵ increases.

does not depend on the dimension of the system, and also that the existence of the two different behaviors is a direct consequence of the density gradient contribution to the heat flux. If $\mu^*(\epsilon)\to 0$ then $s_{\rm long}(k\to\infty)$ is always negative and a finite value of $k^*(\epsilon)$ always exists.

For $\epsilon<\epsilon^*$, the longitudinal shear mode is unstable for all values of k. That means that there are always instabilities with a wavelength covering all the system, independent of the size of the system. Consider next the case of $\epsilon>\epsilon^*$. There is always a value of the wave number, $k^*_{\rm long}(\epsilon)$, such that for $k>k^*_{\rm long}$ all the modes are stable, while for $k<k^*_{\rm long}$ the heating mode is unstable. The function $k^*_{\rm long}(\epsilon)$ is obtained from Eq. (42) by setting $s=0$. The positive root of the equation is

$$k^*_{\rm long}(\epsilon)=\left(\frac{d(1-\epsilon^2)}{2(\kappa^*-\mu^*)}\right)^{1/2}. \tag{44}$$

In Fig. 5 we have plotted this function, also for $d=2$, in the interval $0.65<\epsilon^2<1$, since for $\epsilon\to\epsilon^*$, $k^*_{\rm long}(\epsilon)$ diverges, as discussed above. The function $k^*_{\rm trans}$ is included for compari-

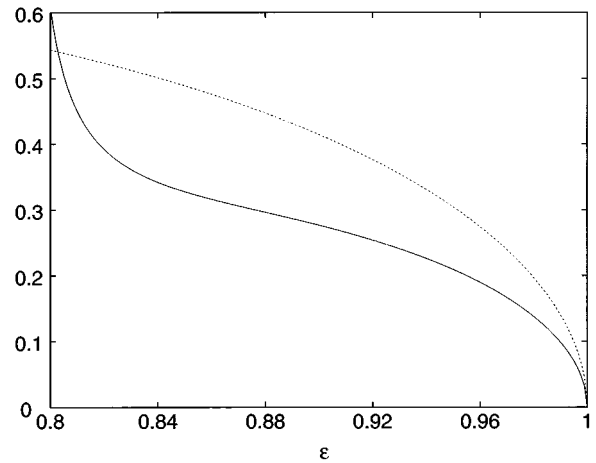

FIG. 5. The critical wave numbers $k^*_{\rm trans}$ (dashed) and $k^*_{\rm long}$ (solid) as functions of ϵ.

son. It is seen that as k is decreased, the first effect which appears is the slow decay of velocity fluctuations associated with the transversal shear modes, except very close to the critical value $k^*_{\rm long}$. As k is decreased further, the longitudinal shear mode becomes also unstable, with the instability growing algebraically in time.

Hydrodynamic modes of a uniform granular medium have been studied recently by McNamara [7], using a fluid-mechanical model. His results are qualitatively similar to ours for values of ϵ very close to one. In particular, he also derives expressions for $k^*_{\rm trans}$ and $k^*_{\rm long}$, which turn out to be proportional to $(1-\epsilon^2)^{1/2}$ as in Eqs. (41) and (44). Nevertheless, the hydrodynamic equations he uses do not contain the term proportional to the gradient of the density in the heat flux and, therefore, they do not predict a critical value of the restitution coefficient. For values of ϵ not very close to one, the behaviors of the modes predicted by our model are more complex than those resulting from his hydrodynamic equations. We refer the reader to his careful and illuminating discussion of the implications of the instability of the homogeneous state.

In a given system, the smallest wave number k_m allowed for a perturbation or fluctuation will be determined by the system geometry and the boundary conditions. We can estimate this value as given by $2\pi/L$, where L is the parameter characterizing the size of the system. Taking into account our choice of dimensionless units, $k_m \simeq v_0/(\zeta_h L)$, which is proportional to the mean free path in the basic homogeneous state. If $k_m > k^*$, the basic homogeneous solution is asymptotically stable, since fluctuations which would lead to a divergent behavior are impossible, while if $k_m < k^*$ the homogeneous state is unstable.

V. CONCLUDING REMARKS

In this paper a model kinetic equation has been proposed to study the dynamics of a gas of particles which collide inelastically. In spite of its apparent simplicity, the model leads to a quite intricate collective behavior of the fluid, and to results which are compatible, at least at a qualitative level, with what has been observed in computer simulation. In particular, it predicts the instability of the uniform cooling state, and reproduces the physical mechanisms which have been found responsible for the formation of clusters in granular media by using other theories. On the other hand, the model can be exactly solved for particular states of the fluid. This allowed us to compute the velocity distribution corresponding to the uniform state, which is needed to obtain the hydrodynamic equations to first order in the gradients (Navier-Stokes approximation). The heat flux is given by the usual Fourier law plus an additional term proportional to the density gradient. This term has been obtained in the study by Lun *et al.* [9] but their analysis is restricted to the quasielastic limit since they have performed the Chapman-Enskog expansion about a local Maxwellian reference state. Since it is proportional to $(1-\epsilon^2)$ it is small at weak dissipation and usually neglected. We are not aware of any other consideration of this density gradient term in the stability analysis of the cooling state.

The broad scope of problems which can be addressed with this model should be emphasized. For instance, it includes the possibility of exploring the microscopic origin of the hydrodynamic instabilities, and also instabilities which are not of hydrodynamic nature. Finally, we mention work in progress to investigate the stability of a state representing uniform shear flow, for which the exact solution of the model kinetic equation also can be obtained. In a different direction, the model can be extended directly to kinetic equations for the dynamics of nonequilibrium fluctuations at one and two times [17].

The BGK model for normal fluids has an underlying basis in the Boltzmann equation. It is well known that the Boltzmann equation is anomalous in one dimension. Thus, although we have developed most of our calculations for arbitrary dimension, its application to the one-dimensional case can be problematic and we do not expect our present model to describe the kind of behavior discussed in Refs. [2,18]. Also, the Boltzmann equation is limited to low density and this is reflected in the quantitative features of our BGK model here as well. We believe that a detailed analysis of the Boltzmann kinetic equation for inelastic collisions will show that the qualitative features of that equation are preserved by the BGK model, both at the hydrodynamic and kinetic levels. This is certainly the case for normal fluids, where the direct Monte Carlo simulation method [19] has been applied to compare results from both the BGK and Boltzmann equations. This method applies even to states far from equilibrium and its accuracy is well established. There should be no difficulty in extending it to the inelastic case [20].

From a more practical point of view, the generalized Chapman-Enskog expansion described here can be applied to an extension of the dense fluid Enskog kinetic theory for a realistic description of granular media. Previous work in this direction has been limited to the quasielastic limit. We plan to discuss the derivation of hydrodynamics from the Enskog equation, and extensions of both the kinetic model and the Monte Carlo simulation method to this dense fluid equation.

ACKNOWLEDGMENTS

The research of J.J.B. was partially supported by Grant No. PB95-0534 from the Dirección General de Investigación Científica y Técnica (Spain). The research of J.W.D. was supported by National Science Foundation Grant No. PHY 9312723. The authors are grateful to Professor M. Erns for a critical reading of the manuscript.

APPENDIX A: ORIGIN OF THE BGK MODEL

The BGK model introduced in Sec. II is intended to capture the most important physical and structural features of the Boltzmann equation. It has been used widely with great success for the case of elastic collisions, and the objective of this appendix is to motivate its relevance for inelastic collisions as well. The Boltzmann equation for this latter case is [20]

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}) f(\mathbf{r},\mathbf{v},t) = J[f,f], \tag{A1}$$

$$J[f,f]\equiv\sigma^2\int d\mathbf{v}_1\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})|\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}|$$
$$\times[\epsilon^{-2}f(\mathbf{r},\mathbf{v}',t)f(\mathbf{r},\mathbf{v}_1',t)$$
$$-f(\mathbf{r},\mathbf{v},t)f(\mathbf{r},\mathbf{v}_1,t)]. \tag{A2}$$

Equation (A2) defines the nonlinear Boltzmann collision operator in terms of the restituting collisions, $\mathbf{v}'=\mathbf{v}-(2\epsilon)^{-1}(1+\epsilon)(\hat{\sigma}\cdot\mathbf{g})\hat{\sigma}$ and $\mathbf{v}_1'=\mathbf{v}_1+(2\epsilon)^{-1}(1+\epsilon)(\hat{\sigma}\cdot\mathbf{g})\hat{\sigma}$. Here, $\hat{\sigma}$ is a unit vector defining the integration over the spherical surface of the colliding pair of particles, σ is the hard sphere diameter, and $\mathbf{g}=\mathbf{v}-\mathbf{v}_1$. It is straightforward to verify the following properties of the collision operator:

$$\int d\mathbf{v}\begin{pmatrix}1\\ \mathbf{v}\\ \frac{1}{2}mv^2\end{pmatrix}J[f,f]=\begin{pmatrix}0\\ \mathbf{0}\\ (1-\epsilon^2)w\end{pmatrix}, \tag{A3}$$

$$w(\mathbf{r},t)=\frac{m\pi\sigma^2}{16}\int d\mathbf{v}\,d\mathbf{v}_1 f(\mathbf{r},\mathbf{v},t)f(\mathbf{r},\mathbf{v}_1,t)|\mathbf{v}-\mathbf{v}_1|^3. \tag{A4}$$

(For simplicity, here and in the remainder of this appendix we limit attention to the case of three dimensions.) With these results, the balance equations (5)–(7) are obtained except with the source term in the temperature equation, $-(1-\epsilon^2)3/2nk_BT\zeta$, replaced with $-(1-\epsilon^2)w$. The BGK model kinetic equation is obtained by using a simpler, more practical collision operator in (A2). In many cases the primary interest is in transport associated with the balance equations, so the conditions given by Eqs. (A3) will be necessary constraints on its choice.

The Boltzmann collision operator has two contributions, representing scattering out of the velocity state $\mathbf{v}$ and scattering into this state. Hence $J[f,f]$ can be written in the suggestive form

$$J[f,f]=-\zeta(\mathbf{v}|f)[f(\mathbf{r},\mathbf{v},t)-h(\mathbf{r},\mathbf{v},t|f)], \tag{A5}$$

with the identifications

$$\zeta(\mathbf{v}|f)=\sigma^2\int d\mathbf{v}_1\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})|\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}|f(\mathbf{r},\mathbf{v}_1,t), \tag{A6}$$

$$h(\mathbf{r},\mathbf{v},t|f)=\zeta(\mathbf{v}|f)^{-1}\sigma^2$$
$$\times\int d\mathbf{v}_1\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})|\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}|\epsilon^{-2}$$
$$\times f(\mathbf{r},\mathbf{v}',t)f(\mathbf{r},\mathbf{v}_1',t). \tag{A7}$$

The notation indicates that both ζ and h are functionals of f. To construct the BGK model we first restrict this functional dependence to occur only through the macroscopic fields, $n(\mathbf{r},t)$, $T(\mathbf{r},t)$, and $\mathbf{u}(\mathbf{r},t)$ as defined by Eqs. (4). The first approximation is to replace $\zeta(\mathbf{v}|f)$ by a velocity-independent function

$$\zeta(\mathbf{v}|f)\to\zeta(n(\mathbf{r},t),T(\mathbf{r},t)). \tag{A8}$$

This means that the collision rate can vary with the local density and temperature of any nonequilibrium state, but that the detailed dependence on the velocity is assumed to be not essential. With this approximation in Eq. (49) the constraints (A3) lead to

$$\int d\mathbf{v}\begin{pmatrix}1\\ \mathbf{v}\end{pmatrix}h(\mathbf{r},\mathbf{v},t)=\begin{pmatrix}n(\mathbf{r},t)\\ n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)\end{pmatrix}, \tag{A9}$$

$$\int d\mathbf{v}\frac{1}{2}mv^2h(\mathbf{r},\mathbf{v},t)=\frac{3}{2}n(\mathbf{r},t)k_BT(\mathbf{r},t)-(1-\epsilon^2)\frac{w}{\zeta}. \tag{A10}$$

The right sides of these equations are functionals of $f(\mathbf{r},\mathbf{v},t)$ through their definitions, Eqs. (4). The second and final assumption of the BGK model is that $h(\mathbf{r},\mathbf{v},t)$ can be replaced by the ''best'' distribution function subject to knowledge of only these constraints, i.e., that which maximizes the information entropy, $S[f]\equiv-\int d\mathbf{v}h(\mathbf{r},\mathbf{v},t|f)\ln h(\mathbf{r},\mathbf{v},t|f)$. A straightforward calculation leads to

$$h(\mathbf{r},\mathbf{v},t|f)\to f_0(\mathbf{r},\mathbf{v},t|f)$$
$$\equiv n(\mathbf{r},t)\left(\frac{m}{2\pi k_BT'(\mathbf{r},t)}\right)^{3/2}$$
$$\times e^{-\{m[\mathbf{v}-\mathbf{u}(\mathbf{r},t)]^2\}/[2k_BT'(\mathbf{r},t)]}, \tag{A11}$$

$$T'(\mathbf{r},t)\equiv T(\mathbf{r},t)\left[1-\frac{2(1-\epsilon^2)w(\mathbf{r},t)}{3nk_BT(\mathbf{r},t)\zeta(\mathbf{r},t)}\right]. \tag{A12}$$

Use of Eqs. (A8) and (A11) in Eq. (A5) defines the BGK kinetic model for the collision operator

$$J_{\mathrm{BGK}}[f,f]\equiv-\zeta(n(\mathbf{r},t),T(\mathbf{r},t))$$
$$\times[f(\mathbf{r},\mathbf{v},t)-f_0(\mathbf{r},\mathbf{v},t|f)]. \tag{A13}$$

It preserves the conditions (A3) by definition and hence implies the same macroscopic balance equations as the Boltzmann equation. In the special case of elastic collisions, $\epsilon=1$, the usual BGK model for the Boltzmann equation results. However, the ''derivation'' described here makes no assumptions specific to elastic collisions. The basic idea is that the macroscopic balance equations and associated macroscopic fields are the most important ingredients of the description. Detailed velocity dependencies of the collision process are assumed to affect only quantitative rather than qualitative dynamics. For elastic collisions, the assumption of maximum information entropy leads directly to the correct stationary solution. For inelastic collisions, detailed balance is violated so no solution to $J[f,f]=0$ occurs. Instead, the homogeneous cooling state occurs from an imbalance between collisions into and out of the velocity state $\mathbf{v}$. This imbalance is preserved in the BGK model through the difference between the temperature $T(\mathbf{r},t)$ associated with the distribution function $f(\mathbf{r},\mathbf{v},t)$ and the effective temperature $T'(\mathbf{r},t)$ associated with the distribution function $h(\mathbf{r},\mathbf{v},t)$.

Dimensional analysis leads to the result

$$w \propto \frac{3}{2} n k_B T \zeta. \tag{A14}$$

In the text we choose the simplest case for which the proportionality constant in this relation is unity.

APPENDIX B: GLOBAL AND LOCAL HOMOGENEOUS SOLUTIONS

In this appendix the derivation of the global homogeneous solution (16) and its extension to the local form (21) is described. It is sufficient to consider the latter since the former follows as a special case. The kinetic equation to lowest order in the uniformity parameter is given by

$$\left(\frac{\partial^{(0)}}{\partial t} + \zeta(t) \right) f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) = \zeta(t) f_0(\mathbf{v},\mathbf{r},t), \tag{B1}$$

where $f_0(\mathbf{v},\mathbf{r},t)$ is defined by Eqs. (2) and (3). The only contribution to the time derivative at lowest order in the uniformity parameter comes from the temperature

$$\frac{\partial^{(0)}}{\partial t} f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) = -(1-\epsilon^2)\zeta(t) T \frac{\partial}{\partial T} \times f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}), \tag{B2}$$

and Eq. (B1) simplifies to

$$\left((1-\epsilon^2) T \frac{\partial}{\partial T} - 1 \right) f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) = -f_0(\mathbf{v},\mathbf{r},t). \tag{B3}$$

It is now straightforward to integrate this equation from $T_0 = T(t_0)$ to $T = T(t)$, assuming a given initial distribution as a function of $T(t_0)$, obtaining

$$f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) = \left(\frac{T}{T_0} \right)^{1/(1-\epsilon^2)} f^{(0)}(\mathbf{v},T_0) + (1-\epsilon^2)^{-1} \int_T^{T_0} \frac{dT'}{T'} \left(\frac{T}{T'} \right)^{1/(1-\epsilon^2)} \times n(\mathbf{r},t)\, \phi(V/\epsilon v_0'), \tag{B4}$$

where the explicit form for f_0 given by Eq. (2) has been used, and v_0' is the thermal velocity as a function of T'. For $T \ll T_0$ the first term on the right side becomes negligible and the solution approaches a universal form independent of the initial conditions. Formally, this universal form is identified by integrating from infinitely high temperature ($T_0 \to \infty$),

$$f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) \to (1-\epsilon^2)^{-1} \int_T^{\infty} \frac{dT'}{T'} \left(\frac{T}{T'} \right)^{1/(1-\epsilon^2)} \times n(\mathbf{r},t)\, \phi(V/\epsilon v_0'). \tag{B5}$$

Finally, a change of variables to integrate over $(T'/T)^{1/2}$ instead of T gives the desired result,

$$f^{(0)}(\mathbf{v}|\{\chi_i(\mathbf{r},t)\}) = \int_1^{\infty} dx\, P(x) n(\mathbf{r},t)\, \phi(V/v_0 \epsilon x), \tag{B6}$$

where $P(x)$ is given by Eq. (17). This is the result (21) used in the main text. The corresponding global solution given by Eq. (16) follows from this local solution for the special case of constant density and vanishing flow velocity.

APPENDIX C: THE FIRST CHAPMAN-ENSKOG APPROXIMATION

Here the generalized Chapman-Enskog expansion is carried out to first order in the uniformity parameter. Substitution of Eqs. (19) and (20) into the kinetic equation and retaining terms of first order in the uniformity parameter gives

$$\frac{\partial^{(1)} f^{(0)}}{\partial t} + \frac{\partial^{(0)} f^{(1)}}{\partial t} + \mathbf{v}.\boldsymbol{\nabla} f^{(0)} = -\zeta f^{(1)}. \tag{C1}$$

The terms involving $f^{(0)}$ are calculated as follows:

$$\left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} \right) f^{(0)} = \frac{\partial f^{(0)}}{\partial n} \left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} \right) n + \cdots, \tag{C2}$$

where the dots on the right side denote corresponding terms from partial derivatives with respect to T and $\mathbf{u}$. The time derivatives of the hydrodynamic fields appearing on the right side have contributions of first order in the uniformity parameter given by Eqs. (5)–(7),

$$\frac{\partial^{(1)} n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{u}) = 0, \tag{C3}$$

$$mn \frac{\partial^{(1)} \mathbf{u}}{\partial t} + mn\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}\cdot\mathbf{P}^{(0)}, \tag{C4}$$

$$\frac{d}{2} n k_B \frac{\partial^{(1)} T}{\partial t} + \frac{d}{2} n k_B \mathbf{u}\cdot\boldsymbol{\nabla} T = -(\boldsymbol{\nabla}\mathbf{u}):\mathbf{P}^{(0)} - \boldsymbol{\nabla}\cdot\mathbf{q}^{(0)}. \tag{C5}$$

The lowest order pressure tensor $\mathbf{P}^{(0)}$ and heat flux $\mathbf{q}^{(0)}$ are calculated directly from Eqs. (8), (9), and (64) with the results

$$\mathbf{P}^{(0)}(\mathbf{r},t) = n(\mathbf{r},t) k_B T(\mathbf{r},t)\mathbf{I}, \quad \mathbf{q}^{(0)}(\mathbf{r},t) = 0. \tag{C6}$$

Use of these results in (C2) gives

$$\left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} \right) f^{(0)} = \mathbf{B}_n\cdot\boldsymbol{\nabla} n + \mathbf{B}_\mathbf{u}:\boldsymbol{\nabla}\mathbf{u} + \mathbf{B}_T\cdot\boldsymbol{\nabla} T, \tag{C7}$$

with the functions $\mathbf{B}_\mathbf{n}$, $\mathbf{B}_\mathbf{u}$, and $\mathbf{B}_\mathbf{u}$ defined by

$$\mathbf{B}_\mathbf{n} = \int_1^{\infty} dx\, P(x)\, \phi(V/v_0\epsilon x) \mathbf{V} \left[1 - \left(\frac{1}{\epsilon^2 x^2} \right)^2 \right], \tag{C8}$$

$$\mathbf{B_u}=\int_1^\infty dx\,P(x)n(\mathbf{r},t)\phi(V/v_0\epsilon x)2(v_0\epsilon x)^{-2}\times(\mathbf{VV}-d^{-1}V^2\mathbf{I}),\tag{C9}$$

$$\mathbf{B_T}=\int_1^\infty dx P(x)n(\mathbf{r},t)\phi(V/v_0\epsilon x)(v_0\epsilon x)^{-2}T^{-1}\mathbf{V}\times\left(V^2-v_0^2-\frac{d}{2}(v_0\epsilon x)^2\right).\tag{C10}$$

With these results Eq. (C1) becomes

$$\left(\frac{\partial^{(0)}}{\partial t}+\zeta\right)f^{(1)}=-(\mathbf{B}_n\cdot\boldsymbol{\nabla}n+\mathbf{B_u}:\nabla\mathbf{u}+\mathbf{B}_T\cdot\boldsymbol{\nabla}T).\tag{C11}$$

The terms occurring on the right side suggest looking for a solution of the form

$$f^{(1)}=\mathbf{A}_n\cdot\boldsymbol{\nabla}n+\mathbf{A_u}:\nabla\mathbf{u}+\mathbf{A}_T\cdot\boldsymbol{\nabla}T.\tag{C12}$$

Substitution of this form into Eq. (C11) and equating coefficients of the independent hydrodynamic gradients gives the equations for $\mathbf{A}_n$, $\mathbf{A_u}$, and $\mathbf{A}_T$,

$$\left((1-\epsilon^2)T\frac{\partial}{\partial T}-1\right)\mathbf{A}_n+(1-\epsilon^2)\frac{T}{n}\mathbf{A}_T=\zeta^{-1}\mathbf{B}_n,\tag{C13}$$

$$\left((1-\epsilon^2)T\frac{\partial}{\partial T}-1\right)\mathbf{A_u}=\zeta^{-1}\mathbf{B_u},\tag{C14}$$

$$\left((1-\epsilon^2)T\frac{\partial}{\partial T}+\frac{1}{2}(1-3\epsilon^2)\right)\mathbf{A}_T=\zeta^{-1}\mathbf{B}_T.\tag{C15}$$

It is a straightforward but lengthy calculation to construct the solution to these equations by integrating from an infinite initial temperature, as was done in Appendix B. Here, we are interested primarily in the heat and momentum fluxes defined by Eqs. (8) and (9). Use of the form (C12) leads to the Navier-Stokes order fluxes given by Eqs. (23) and (24), with the shear viscosity, thermal conductivity, and μ defined by

$$\eta=-\int d\mathbf{v}\,mV_xV_y(\mathbf{A_u})_{xy},\quad \kappa=-\frac{1}{d}\int d\mathbf{v}\frac{1}{2}mV^2\mathbf{V}\cdot\mathbf{A}_T,$$

$$\mu=-\frac{1}{d}\int d\mathbf{v}\frac{1}{2}mV^2\mathbf{V}\cdot\mathbf{A}_n.\tag{C16}$$

The above equations for $\mathbf{A}_n$, $\mathbf{A_u}$, and $\mathbf{A}_T$ now can be used to obtain the corresponding equations for the transport coefficients. They read

$$\left((1-\epsilon^2)T\frac{\partial}{\partial T}-1\right)\mu+(1-\epsilon^2)\frac{T}{n}\kappa=-\frac{3mv_0^4}{8d\zeta}(d+2)\frac{(1-\epsilon^2)^2}{2\epsilon^2-1},\tag{C17}$$

$$\left((1-\epsilon^2)T\frac{\partial}{\partial T}-1\right)\eta=-n\frac{mv_0^2}{2\zeta},\tag{C18}$$

$$\left((1-\epsilon^2)T\frac{\partial}{\partial T}+\frac{1}{2}(1-3\epsilon^2)\right)\kappa=-n\frac{3mv_0^4}{8dT\zeta}(d+2)\frac{(2\epsilon^4-2\epsilon^2+1)}{2\epsilon^2-1}.\tag{C19}$$

The right hand sides of (C17) and (C19) result from velocity moments of degree 4. According to the discussion following (16) these integrals exist only if $p(\epsilon)>4$ or, equivalently, $\epsilon^2>1/2$. The existence of hydrodynamics for our kinetic model at Navier-Stokes order is therefore limited as well by this condition.

Equations (C18) and (C19) can be integrated directly to obtain η and κ. Then, use of this result for κ in (C17) gives μ. In this way Eqs. (25)–(27) are obtained.

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